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Reverse inequalities involving two relative operator entropies and two relative entropies

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Abstract. We shall discuss relation among Tsallis relative operator entropy $T_p(A, B)$, the relative operator entropy $\hat{S}(A|B)$ by J.I.Fujii-Kamei, the Tsallis relative entropy $D_p(A||B)$ by Furuichi-Yanagi-Kuriyama and the Umegaki relative entropy $S(A, B)$. We show the following result: *Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$ and $p \in (0, 1]$. Then the following inequalities hold:*

$$\left(\frac{1 - K(p)}{p}\right)(\text{Tr}[A])^{1-p}(\text{Tr}[B])^p + D_p(A||B) \geq -\text{Tr}[T_p(A|B)] \geq D_p(A||B)$$

where $K(p)$ is the generalized Kantorovich constant defined by

$$K(p) = \frac{(h^p - h)}{(p-1)(h-1)} \left(\frac{(p-1)(h^p - 1)}{p(h^p - h)} \right)^p$$

and the first inequality is the reverse one of the second known inequality, in particular

$$\log S(1)\text{Tr}[A] + S(A, B) \geq -\text{Tr}[\hat{S}(A|B)] \geq S(A, B)$$

where $S(1)$ is the Specht ratio defined by

$$S(1) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$$

and the first inequality is the reverse one of the second known inequality.

It is known that $K(p) \in (0, 1]$ for $p \in (0, 1]$ and $S(1) > 1$.

§1. Introduction

A capital letter means a $n \times n$ complex matrix and $\text{Tr}[X]$ means the trace on the matrix X . A matrix X is said to be *strictly positive definite* if X is positive definite and invertible (denoted by $X > 0$). Let A and B be strictly positive definite matrices. Umegaki relative entropy $S(A, B)$ in [11] is defined by

$$(1.1) \quad S(A, B) = \text{Tr}[A(\log A - \log B)]$$

and the relative operator entropy $\hat{S}(A|B)$ in [2] is defined by

$$(1.2) \quad \hat{S}(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

as an extension of [10]. Very recently, Tsallis relative operator entropy $T_p(A|B)$ in Yanagi-Kuriyama-Furuichi [12] is defined by

$$(1.3) \quad T_p(A|B) = \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^p A^{\frac{1}{2}} - A}{p}$$

for $p \in (0, 1]$ and also Tsallis relative entropy $D_p(A||B)$ in Furuichi-Yanagi-Kuriyama [4] is defined by

$$(1.4) \quad D_p(A||B) = \frac{\text{Tr}[A] - \text{Tr}[A^{1-p} B^p]}{p}$$

for $p \in (0, 1]$. Next we shall state the following results on $-\text{Tr}[T_p(A|B)]$, $D_p(A||B)$, $-\text{Tr}[\hat{S}(A|B)]$ and $S(A, B)$.

Theorem A. (Generalized Peierls-Bogoliubov inequality [4]) *Let $A, B > 0$ and also let $p \in (0, 1]$. Then the following inequality holds:*

$$(1.5) \quad D_p(A||B) \geq \frac{\text{Tr}[A] - (\text{Tr}[A])^{1-p}(\text{Tr}[B])^p}{p}.$$

Theorem B. *Let $A, B > 0$. The following inequality holds:*

$$(1.6) \quad -\text{Tr}[T_p(A|B)] \geq D_p(A||B) \quad \text{for } p \in (0, 1]$$

and

$$(1.7) \quad -\text{Tr}[\hat{S}(A|B)] \geq S(A, B).$$

Theorem C. [4] *The following properties hold:*

$$(1.8) \quad \lim_{p \rightarrow 0} T_p(A|B) = \hat{S}(A|B)$$

and

$$(1.9) \quad \lim_{p \rightarrow 0} D_p(A||B) = S(A, B).$$

Let $h > 1$. The generalized Kantorovich constant $K(p)$ is defined by

$$(1.10) \quad K(p) = \frac{(h^p - h)}{(p-1)(h-1)} \left(\frac{(p-1)(h^p - 1)}{p(h^p - h)} \right)^p$$

for any real number p and it is known that $K(p) \in (0, 1]$ for $p \in [0, 1]$. Also $S(p)$ is defined by

$$(1.11) \quad S(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}}$$

for any real number p . In particular $S(1) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is said to be *the Specht ratio* and $S(1) > 1$ is well known. We state the known results on the generalized Kantorovich constant $K(p)$ and Specht ratio $S(1)$ (for example, [6]). Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities (1.12), (1.13) and (1.14) hold for every unit vector x and (1.12) is equivalent to (1.13):

$$(1.12) \quad K(p)(Ax, x)^p \geq (A^p x, x) \geq (Ax, x)^p \quad \text{for any } p > 1 \text{ or any } p < 0.$$

$$(1.13) \quad (Ax, x)^p \geq (A^p x, x) \geq K(p)(Ax, x)^p \quad \text{for any } 1 \geq p > 0.$$

$$(1.14) \quad S(1)\Delta_x(A) \geq (Ax, x) \geq \Delta_x(A).$$

where the determinant $\Delta_x(A)$ for strictly positive operator A at a unit vector x is defined by $\Delta_x(A) = \exp\langle (\log A)x, x \rangle$ and (1.14) is shown in [3].

Theorem D. (i) $K(p)$ is symmetric with respect to $p = \frac{1}{2}$ and $K(p)$ is an increasing function of p for $p \geq \frac{1}{2}$, and, $K(p)$ is a decreasing function of p for $p \leq \frac{1}{2}$, and $K(0) = K(1) = 1$.

$$(ii) \quad K(p) \geq 1 \text{ for } p \geq 1 \text{ or } p \leq 0, \text{ and } 1 \geq K(p) \geq \frac{2h^{\frac{1}{2}}}{h^{\frac{1}{2}}+1} \text{ for } p \in [0, 1].$$

(iii) $S(p)$ is symmetric with respect to $p = 0$ and $S(p)$ is an increasing function of p for $p \geq 0$, and, $S(p)$ is a decreasing function of p for $p \leq 0$ and $S(0) = 1$.

$$(iv) \quad S(1) = e^{K'(1)} = e^{-K'(0)}.$$

(iv) of Theorem D is shown in [Proposition 1, 5] and (i), (ii) and (iii) are shown in [6].

For two strictly positive definite matrices A, B and $p \in [0, 1]$, p -power mean $A \sharp_p B$ is defined by

$$A \sharp_p B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^p A^{\frac{1}{2}}$$

and we remark that $A \sharp_p B = A^{1-p} B^p$ if A commutes with B .

§2. Reverse inequalities involving $-\text{Tr}[T_p(A|B)]$, $D_p(A||B)$, $-\text{Tr}[\hat{S}(A|B)]$ and $S(A, B)$

We shall show the following reverse inequalities involving $-\text{Tr}[T_p(A|B)]$, $D_p(A||B)$, $-\text{Tr}[\hat{S}(A|B)]$ and $S(A, B)$.

Theorem 1. Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$ and $p \in (0, 1]$. Then the following inequalities hold:

$$\begin{aligned}
(2.1) \quad & \left(\frac{1 - K(p)}{p} \right) (\text{Tr}[A])^{1-p} (\text{Tr}[B])^p + D_p(A||B) \\
& \geq -\text{Tr}[T_p(A|B)] \\
& \geq D_p(A||B)
\end{aligned}$$

where $K(p)$ is the generalized Kantorovich constant defined in (1.10) and the first inequality is the reverse one of the second inequality.

Corollary 2. Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold:

$$\begin{aligned}
(2.2) \quad & \log S(1) \text{Tr}[A] + S(A, B) \\
& \geq -\text{Tr}[\hat{S}(A|B)] \\
& \geq S(A, B)
\end{aligned}$$

where $S(1)$ is the Specht ratio defined in (1.11) and the first inequality is the reverse one of the second inequality.

We remark that the second inequality in (2.1) is known in (1.6) of Theorem B and also the second inequality in (2.2) is well known in (1.7) of Theorem B. Also we remark that $\frac{1 - K(p)}{p} \geq 0$ in Theorem 1 by (ii) of Theorem D.

Remark. By using (iv) of Theorem D and Theorem C, (2.1) of Theorem 1 implies (2.2) of Corollary 2. By the same way, It is interesting to point out that by also using (iv) of Theorem D, (1.13) implies (1.14) as follows:

$$(Ax, x) \geq (A^p x, x)^{\frac{1}{p}} \geq K(p)^{\frac{1}{p}} (Ax, x) \quad \text{for any } 1 \geq p > 0.$$

and it is easily verified that $\lim_{p \rightarrow 0} (A^p x, x)^{\frac{1}{p}} = \Delta_x(A)$ and $\lim_{p \rightarrow 0} K(p)^{\frac{-1}{p}} = S(1)$ by (iv) of Theorem D, so that (1.13) implies (1.14).

At the end of this remark, we would like to emphasize that $\lim_{p \rightarrow 0} \frac{1 - K(p)}{p} = \log S(1)$ of (3.4) in the step (2.1) \implies (2.2) and $\lim_{p \rightarrow 0} K(p)^{\frac{-1}{p}} = S(1)$ in the step (1.13) \implies (1.14) are both derived from $S(1) = e^{K'(1)} = e^{-K'(0)}$ by (iv) of Theorem D.

The results in this paper will appear elsewhere.

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